REFINED CONFIGURATION RESULTS FOR
EXTREMAL TYPE II LATTICES OF RANKS 40 AND 80

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ABSTRACT. We show that, if L is an extremal Type II lattice of rank 40 or 80, then L is generated by its vectors of norm $\min(L) + 2$. This sharpens earlier results of Ozeki, and the second author and Abel, which showed that such lattices L are generated by their vectors of norms $\min(L)$ and $\min(L) + 2$.

1. INTRODUCTION

1.1. **Preliminaries.** A *lattice* of *rank* n is a free $\mathbb{Z}$-module of rank n equipped with a positive-definite inner product $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{R}$. For $x \in L$, we call $\langle x, x \rangle$ the *norm* of x. If $\langle x, x' \rangle \in \mathbb{Z}$ for all $x, x' \in L$, then we say that L is *integral*. The lattice L is called *even* if every vector of L has an even integer norm, i.e. if $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in L$. The well-known parallelogram identity,

$$(1) \qquad 2\langle x, x' \rangle = \langle x + x', x + x' \rangle - \langle x, x \rangle - \langle x', x' \rangle,$$

shows that an even lattice must be integral.

The *dual* of L, denoted L^*, is the lattice defined by $L^* = \{x' \in L \otimes \mathbb{R} : \langle x, x' \rangle \in \mathbb{Z} \text{ for all } x \in L\}$. If $L = L^*$ then L is said to be *self-dual*. A self-dual lattice is said to be of *Type II* if it is even, and of *Type I* otherwise.

The rank of a Type II lattice must be divisible by 8 (see [Ser73, p. 53 (Cor. 2)] and [Ser73, p. 109]). Mallows, Odlyzko, and Sloane [MOS75] (see also [CS99, p. 194]) used theta functions and modular forms to show that the *minimal nonzero norm* (or just *minimal norm*) $\min(L)$ of vectors in a Type II lattice L of rank n satisfies the upper bound $\min(L) \le 2\lfloor n/24 \rfloor + 2$. The lattice L is called *extremal* if it attains equality in this bound, that is, if

$$\min(L) = 2\lfloor n/24 \rfloor + 2.$$

1.2. **Configuration Results for Extremal Type II Lattices.** Venkov [Ven84] and Ozeki [Oze86a] showed that if L is an extremal Type II lattice of rank 32 then L is generated by its vectors of minimal norm. Additionally, Ozeki [Oze86b] proved an analogous result for extremal Type II lattices of rank 48. Recently, the second author [Kom09] extended the methods of Ozeki [Oze86b] to obtain analogous results for extremal Type II lattices of ranks 56, 72, and 96.

The following similar but weaker result has been obtained for extremal Type II lattices of ranks 40, 80, and 120.

Theorem 1. *If L is an extremal Type II lattice of rank* 40*,* 80*, or* 120*, then L is generated by its vectors of norms* $\min(L)$ *and* $\min(L) + 2$.

The rank 40 case of this result is due to Ozeki [Oze89], while the cases of ranks 80 and 120 are due to the second author and Abel [KA08]. Ozeki [Oze89] noted that the rank 40 case of Theorem 1 is sharp, in the sense that there exist extremal Type II lattices of rank 40 that are not generated by their vectors of minimal norm.

In this note, we prove the following refinement of Theorem 1.

Theorem 2. *If L is an extremal Type II lattice of rank* 40 *or* 80*, then L is generated by its vectors of norm* $\min(L) + 2$.

2. The Proof of Theorem 2

Let L be an extremal Type II lattice of rank n, and let $m_0 = \min(L)$. Adopting the notation of [KA08], we write $\Lambda_j(L) = \{x \in L : \langle x, x \rangle = j\}$ for the set of norm-j vectors of L, and denote by $\mathcal{L}_j(L)$ the lattice generated by $\Lambda_j(L)$. Also following [KA08], we define, for $\dot{x} \in \mathbb{R}^n$,

$$M_j(L; \dot{x}) = |\{x \in \Lambda_{m_0+2}(L) : \langle \dot{x}, x \rangle = j\}|,$$

noting that $M_{-j}(L; \dot{x}) = M_j(L; \dot{x})$ for any $\dot{x}$ and j.

We recall that Venkov [Ven01] proved using weighted theta functions and modular forms that, for each $j > 0$ such that $\Lambda_j(L) \neq \emptyset$, the shell $\Lambda_j(L)$ is a spherical $(t\frac{1}{2})$-design, where t is 11, 7, or 3 according as n is congruent to 0, 8, or 16 modulo 24. This means that

$$\sum_{x \in \Lambda_j(L)} P(x) = 0 \tag{2}$$

holds for all spherical harmonic polynomials P of degree s with $1 \leq s \leq t$ as well as $s = t + 3$. We shall use (2) with $P(x) = P_{s;\dot{x}}(x)$ for s even and $\dot{x} \in \mathbb{R}^n$, where $P_{d;\dot{x}}$ is the zonal spherical harmonic polynomial of degree d (see [Vil68]).

Now take $n = 40$ or $n = 80$, so that $m_0 = n/10$ and $t = m_0 - 1$. Using the modularity of the theta function of L, we calculate that $|\Lambda_{m_0+2}(L)|$ is 87859200 for $n = 40$, and 7541401190400 for $n = 80$; in particular $\Lambda_{m_0+2}(L) \neq \emptyset$. Let S_n be the set of $m_0/2$ even values of s for which we can apply (2) to $P_{s;\dot{x}}$; that is, $S_{40} = \{2, 6\}$ and $S_{80} = \{2, 4, 6, 10\}$.

We now proceed to prove Theorem 2.

Proof of Theorem 2. By Theorem 1, it suffices to show that every vector in $\Lambda_{m_0}(L)$ is contained in $\mathcal{L}_{m_0+2}(L)$. We therefore suppose that there is some $\dot{x} \in \Lambda_{m_0}(L)$ not in $\mathcal{L}_{m_0+2}(L)$, seeking a contradiction.

For any $x \in \Lambda_{m_0+2}(L)$, we must have

$$|\langle x, \dot{x} \rangle| \in \left\{0, 1, \ldots, \frac{m_0}{2} - 1, \frac{m_0}{2} + 1\right\}. \tag{3}$$

Indeed, if $\langle \dot{x}, \pm x \rangle > \frac{m_0}{2} + 1$, then $[\dot{x}]$ contains a nonzero vector $x \mp \dot{x}$ of norm

$$\langle x \mp \dot{x}, x \mp \dot{x} \rangle = \langle x, x \rangle \mp 2\langle x, \dot{x} \rangle + \langle \dot{x}, \dot{x} \rangle < \langle \dot{x}, \dot{x} \rangle = m_0$$

by (1), contradicting the extremality of L. Furthermore, we may assume $\langle x, \dot{x} \rangle \neq \pm m_0/2$, else we would have $x \mp \dot{x} \in \Lambda_{m_0+2}(L)$ by (1), whence $x = \pm \dot{x} + (x \mp \dot{x}) \in \mathcal{L}_{m_0+2}(L)$ follows. That is, we may assume $M_{m_0/2}(L; \dot{x}) = 0$.

Now, when we take $P = P_{s;\dot{x}}$ in (2), we obtain $P(x) = Q(\langle x, \dot{x} \rangle)$ for some even polynomial Q in one variable. For any such polynomial Q, we have

$$\sum_{x \in \Lambda_{m_0+2}(L)} Q(\langle x, \dot{x} \rangle) = Q(0) \cdot M_0(L; \dot{x}) + 2 \sum_{j=1}^{(m_0/2)+1} Q(j) \cdot M_j(L; \dot{x})$$

$$= Q(0) \cdot M_0(L; \dot{x}) + 2 \sum_{j=1}^{(m_0/2)-1} Q(j) \cdot M_j(L; \dot{x}) + 2Q\left(\frac{m_0}{2} + 1\right) \cdot M_{(m_0/2)+1}(L; \dot{x}).$$

Letting s vary over S_n, we obtain $m_0/2$ homogeneous linear equations in the $(m_0/2) + 1$ variables

$$M_0(L; \dot{x}), \ldots, M_{(m_0/2)-1}(L; \dot{x}), M_{(m_0/2)+1}(L; \dot{x}).$$

In each of our cases $n = 40$ and $n = 80$, we find that these equations are linearly independent and thus have a unique solution up to scaling. We compute that for $n = 40$ the $M_j(L; \dot{x})$ for $j = 0, 1, 3$ are proportional to

$$4688, \ 4293, \ -37,$$

while for $n = 80$ the $M_j(L; \dot{x})$ for $j = 0, 1, 2, 3, 5$ are proportional to

$$5661456, \ 3946750, \ 711000, \ 88875, \ -553.$$

But this is impossible because $M_j(L; \dot{x}) \geq 0$ for all j by definition, and if every $M_j(L; \dot{x})$ vanished then we would have $\Lambda_{m_0+2}(L) = \emptyset$. $\qquad\square$

3. Remarks

It is not known whether the conclusion of Theorem 2 holds for L extremal and Type II of rank 120. We cannot directly adapt our approach to this case, because when $n = 120$ (and $m_0 = 12$) the system used in the proof of Theorem 2 yields nonnegative ratios among the values $M_j(L; \dot{x})$ for $j \in \{0, 1, 2, 3, 4, 5, 7\}$. However, the first author [Elk09] has obtained a different refinement of Theorem 1 in the rank-120 case: any extremal Type II lattice of rank 120 is generated by its vectors of minimal norm.

Acknowledgements

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